Iterative Methods for Linear System

Recall: $A\vec{x} = \vec{b}$

$$\vec{x}^{k+1} = M\vec{x} + \vec{f}$$

$$\vec{e^k} = M^k \vec{e}^0$$

Converge iff

$\vec{e}^k \to 0$ or $\|\vec{e}^k\| \to 0$ as $k \to \infty$.

Note $\|\vec{e}^k\| = \|M^k \vec{e}^0\|$

$\leq \rho(M)^k \|\vec{e}_0\|$ as $k \to \infty$

$\therefore \|\vec{e}^k\| \to 0$ iff $\rho(M) < 1$,

converge faster for smaller $\rho(M)$.

HW Q(5).

(a), convergence iff $\frac{\alpha-1}{\lambda_1} < t < \frac{\alpha+1}{\lambda_n}$

If $\frac{\alpha-1}{\lambda_1} > \frac{\alpha+1}{\lambda_n}$, no t will converge.

(b), Even if no t will converge,

$t = \frac{2\alpha}{\lambda_1+\lambda_n}$ will still gives

the smallest spectral radius.

SOR Methods

$$A\vec{x} = \vec{b}.$$

$$\vec{x}^{k+1} = \underbrace{\left(L+\tfrac{1}{\omega}D\right)^{-1}\left(\tfrac{1}{\omega}D-(D+U)\right)}_{M_{SOR}}\vec{x}^{k} + \left(L+\tfrac{1}{\omega}D\right)^{-1}\vec{b}.$$

Difficult to compute spectral radius in general.

Convergence Thm:

(1) SDD + $0 < \omega \le 1$ (SOR convergence)

(2). D. Young (SOR convergence, optimal ω and rate).

(i) $0 < \omega < 2$

(ii). M_J has only real eigenvalues.

(iii) $\beta := \rho(M_J) < 1$

(iv) A consistently ordered.

$\Rightarrow$ SOR converge.

Moreover, $\omega_{opt} = \dfrac{2}{1+\sqrt{1-\beta^2}}$,

$$\rho(M_{SOR,\omega_{opt}}) = \omega_{opt} - 1$$

(3). Householder - John. (Not only SOR)

A, $N^* + N - A$ self-adjoint, positive definite

$\Rightarrow$ $N\vec{x}^{k+1} = P\vec{x}^{k} + \vec{b}$

converges.

Example

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

(1) SDD $\Rightarrow$ converge if $0 < \omega \leq 1$

(2). $$M_j = \begin{bmatrix} 0 & -1/4 \\ -1/4 & 0 \end{bmatrix}$$

$$\Rightarrow \lambda = 1/4 \text{ or } -1/4 \in \mathbb{R}$$

$$\Rightarrow \rho(M_j) = 1/4 = \beta$$

$$W_{opt} = \frac{2}{1 + \sqrt{1 - \beta^2}}$$

$$= \frac{2}{1 + \sqrt{15}/4} = 1.0161\ldots$$

$$\rho(M_{SOR}, W_{opt}) = 0.0161\ldots$$

Thm

$A \in \mathbb{R}^{n \times n}$, symmetric, SDD, $+$ve diagonal

$\Rightarrow$ A symmetric positive definite (SPD).

pf.

Symmetric $\Rightarrow$ A has n real eigenvalues.

SDD $+$ Gershgorin Circle Thm $+$ $+$ve diagonal

$\Rightarrow$ all eigenvalues has $+$ve real part.

$\therefore$ A has n $+$ve eigenvalues. $\square$

Rmk

If instead,

A is diagonal dominant (not strictly)

and non-negative diagonal,

Then

A is symmetric positive semidefinite

(3). $N\vec{x}^{k+1} = P\vec{x}^{k} + \vec{b}$, $A = N - P$

SOR: $N = (L + \frac{1}{\omega} D)$

$P = (\frac{1}{\omega} - D - U)$

$A^* = \overline{A^T} = A$, A self adjoint.

Also, A is SPD.

$N^* + N - A$

$= (L^T + \frac{1}{\omega} D) + (L + \frac{1}{\omega} D) - (L + D - U)$

$L = U^T$, $= (\frac{2}{\omega} - 1) D = (\frac{2}{\omega} - 1) \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

SPD when $0 < w < 2$.

Householder - John

$\Rightarrow$ SOR converges when $0 < w < 2$.

Consider $\begin{bmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & 1 & 4 \end{bmatrix}$

(1). SDD $\Rightarrow$ $0 < w \le 1$ converge.

(2). $M_J = \begin{bmatrix} 0 & -1/4 & & & \\ -1/4 & 0 & -1/4 & & \\ & & \ddots & & \\ & & & & -1/4 \\ & & & -1/4 & 0 \end{bmatrix}$

M_J real symmetric

$\Rightarrow$ M_J has only real eigenvalues.

We cannot calculate $\rho(M_J)$ directly,

but we can estimate it.

Gershgorin Circle Thm

$\Rightarrow$ $\beta = \rho(M_J) \le 1/2$

$\therefore$ D. Young $\Rightarrow$ SOR converges for $0 < w < 2$.

$$1 \le W_{opt} = \frac{2}{1 + \sqrt{1 - \beta^2}} \le \frac{2}{1 + \sqrt{1 - (1/2)^2}}$$

$\Rightarrow$ $1 \le W_{opt} \le 1.0718$

and $\rho(M_{SOR, W_{opt}}) \le 0.0718$

Rmk In fact, if $n=6$,

$\rho(M_J) \approx 0.45$

About 6 times faster if we choose $w = 1.0718$

(3). For using Householder - John Thm,

Exactly the same with previous example.

Exercise

For $A\vec{x} = \vec{b}$,

$$A = \begin{bmatrix} 7 & 0 & -2 & & & & \\ 0 & 7 & 0 & -2 & & & \\ -2 & 0 & 7 & 0 & -2 & & \\ & & & \ddots & & & \\ & & & & & & -2 \\ & & & & & 7 & 0 \\ & & & & -2 & 0 & 7 \end{bmatrix}$$

Construct an iterative scheme to solve the Linear System with rate of convergence < 0.1.

Exercise Solution

$$M_J = \begin{bmatrix} 0 & 0 & 2/7 & & & \\ 0 & 0 & 0 & 2/7 & & \\ 2/7 & 0 & 0 & 0 & 2/7 & \\ & & & & \ddots & \\ & & & 2/7 & 0 & 0 \end{bmatrix}$$

is symmetric.

$\Rightarrow$ all eigenvalues are in $\mathbb{R}$.

Gershgorin Circle Theorem.

$\Rightarrow \quad \lambda \in \overline{B_0(4/7)} \quad \forall \lambda$ eigenvalue of M_J.

$\Rightarrow \quad \beta = \rho(M_J) \leq 4/7$.

$$A = \begin{bmatrix} \boxed{\begin{matrix} 7 & 0 \\ 0 & 7 \end{matrix}} & \boxed{\begin{matrix} -2 & \\ 0 & -2 \end{matrix}} & & & \\ \boxed{\begin{matrix} -2 & 0 \\ & -2 \end{matrix}} & \boxed{\begin{matrix} 7 & 0 \\ 0 & 7 \end{matrix}} & \boxed{\begin{matrix} -2 & \\ 0 & -2 \end{matrix}} & & \\ & & \ddots & & \\ & & & \boxed{\begin{matrix} -2 & 0 \\ & -2 \end{matrix}} & \boxed{\begin{matrix} 7 & 0 \\ 0 & 7 \end{matrix}} \end{bmatrix}$$

is block tridiagonal and hence consistently ordered.

$\therefore$ D. Young's Thm

$\Rightarrow$ SOR converges for $0 < \omega < 2$

and $$\omega_{opt} \le \frac{2}{1 + \sqrt{1 - \beta^2}}$$

$$= 1.0985$$

$$\rho(M_{SOR, \omega_{opt}}) \le 0.0985.$$

$\therefore$ choose $\omega = 1.0985$,

we have rate of convergence

$$= \rho(M_{SOR, \omega}) = 0.0985.$$